THE DEHN FUNCTIONS OF $Out(F_n)$ AND $Aut(F_n)$

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Abstract. For $n \geq 3$, the Dehn functions of $Out(F_n)$ and $Aut(F_n)$ are exponential. Hatcher and Vogtmann proved that they are at most exponential, and the complementary lower bound in the case $n = 3$ was established by Bridson and Vogtmann. Handel and Mosher completed the proof by reducing the lower bound for $n > 4$ to the case $n = 3$. In this note we give a shorter, more direct proof of this last reduction.

Dehn functions provide upper bounds on the complexity of the word problem in finitely presented groups. They are examples of filling functions: if a group G acts properly and cocompactly on a simplicial complex X, then the Dehn function of G is asymptotically equivalent to the function that provides the optimal upper bound on the area of least-area discs in X, where the bound is expressed as a function of the length of the boundary of the disc. This article is concerned with the Dehn functions of automorphism groups of finitely-generated free groups.

Much of the contemporary study of $Out(F_n)$ and $Aut(F_n)$ is based on the deep analogy between these groups, mapping class groups, and lattices in semisimple Lie groups, particularly $\mathrm{SL}(n, \mathbb{Z})$. The Dehn functions of mapping class groups are quadratic [9], as is the Dehn function of $\mathrm{SL}(n, \mathbb{Z})$ if $n \geq 5$ (see [10]). In contrast, Epstein *et al.* [6] proved that the Dehn function of $\mathrm{SL}(3, \mathbb{Z})$ is exponential. Building on their result, we proved in [3] that $Aut(F_3)$ and $Out(F_3)$ also have exponential Dehn functions. Hatcher and Vogtmann [8] established an exponential upper bound on the Dehn function of $Aut(F_n)$ and $Out(F_n)$ for all $n \geq 3$. The comparison with $\mathrm{SL}(n, \mathbb{Z})$ might lead one to suspect that this last result is not optimal for large n, but recent work of Handel and Mosher [7] shows that in fact it is: they establish an exponential lower bound by using their general results on quasi-retractions to reduce to the case $n = 3$.

Theorem. *For $n \geq 3$, the Dehn functions of $Aut(F_n)$ and $Out(F_n)$ are exponential.*

This theorem answers Questions 35 and 37 of [4].

We learned the contents of [7] from Lee Mosher at Luminy in June 2010 and realized that one can also reduce the Theorem to the case $n = 3$ using a simple observation about natural maps between different-rank Outer spaces and Auter spaces (Lemma 3). The purpose of this note is record this observation and the resulting proof of the Theorem.

1991 *Mathematics Subject Classification.* 20F65, 20F28, 53C24, 57S25.

Key words and phrases. Automorphism groups of free groups, Dehn functions.

Bridson is supported by an EPSRC Senior Fellowship. Vogtmann is supported by NSF grant DMS-0204185.

1. Definitions. Let A be a 1-connected simplicial complex. We consider simplicial loops $\ell\colon S \to A^{(1)}$, where S is a simplicial subdivision of the circle. A *simplicial filling* of ℓ is a simplicial map $L\colon D \to A^{(2)}$, where D is a triangulation of the 2-disc and $L|_{\partial D} = \ell$. Such fillings always exist, by simplicial approximation. The filling area of ℓ, denoted $\mathrm{Area}_A(\ell)$, is the least number of triangles in the domain of any simplicial filling of ℓ. The *Dehn function*[1] of A is the least function $\delta_A\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{Area}_A(\ell) \leq \delta_A(n)$ for all loops of length $\leq n$ in $A^{(1)}$. The Dehn function of a finitely presented group G is the Dehn function of any 1-connected 2-complex on which G acts simplicially with finite stabilizers and compact quotient. This is well-defined up to the following equivalence relation: functions $f, g\colon \mathbb{N} \to \mathbb{N}$ are equivalent if $f \preceq g$ and $g \preceq f$, where $f \preceq g$ means that there is a constant $a > 1$ such that $f(n) \leq a\, g(an + a) + an + a$. The Dehn function can be interpreted as a measure of the complexity of the word problem for G — see [2].

Lemma 1. *If A and B are 1-connected simplicial complexes, $F\colon A \to B$ is a simplicial map, and ℓ is a loop in the 1-skeleton of A, then* $\mathrm{Area}_A(\ell) \geq \mathrm{Area}_B(F \circ \ell)$.

Proof. If $L\colon D \to A$ is a simplicial filling of ℓ, then $F \circ L$ is a simplicial filling of $F \circ \ell$, with the same number of triangles in the domain D. □

Corollary. *Let A, B and C be 1-connected simplicial complexes with simplicial maps $A \to B \to C$. Let ℓ_n be a sequence of simplicial loops in A whose length is bounded above by a linear function of n, let $\overline{\ell}_n$ be the image loops in C and let $\alpha(n) = \mathrm{Area}_C(\overline{\ell}_n)$. Then the Dehn function of B satisfies $\delta_B(n) \succeq \alpha(n)$.*

Proof. This follows from Lemma 1 together with the observation that a simplicial map does not increase the length of any loop in the 1-skeleton. □

2. Simplicial complexes associated to $Out(F_n)$ and $Aut(F_n)$. Let K_n denote the spine of Outer space, as defined in [5], and L_n the spine of Auter space, as defined in [8]. These are contractible simplicial complexes with cocompact proper actions by $Out(F_n)$ and $Aut(F_n)$ respectively, so we may use them to compute the Dehn functions for these groups.

Recall from [5] that a *marked graph* is a finite metric graph Γ together with a homotopy equivalence $g\colon R_n \to \Gamma$, where R_n is a fixed graph with one vertex and n loops. A vertex of K_n can be represented either as a marked graph (g, Γ) with all vertices of valence at least three, or as a free minimal action of F_n on a simplicial tree (namely the universal cover of Γ). A vertex of L_n has the same descriptions except that there is a chosen basepoint in the marked graph (respected by the marking) or in the simplicial tree. Note that we allow marked graphs to have separating edges. Both K_n and L_n are flag complexes, so to define them it suffices to describe what it means for vertices to be adjacent. In the marked-graph description, vertices of K_n (or L_n) are adjacent if one can be obtained from the other by a forest collapse (i.e. collapsing each component of a forest to a point).

[1] The standard definition of area and Dehn function are phrased in terms of singular discs, but this version is $\simeq$ equivalent.

3. Three Natural Maps. There is a *forgetful map* $\phi_n \colon L_n \to K_n$ which simply forgets the basepoint; this map is simplicial.

Let $m < n$. We fix an ordered basis for F_n, identify F_m with the subgroup generated by the first m elements of the basis, and identify $Aut(F_m)$ with the subgroup of $Aut(F_n)$ that fixes the last $n-m$ basis elements. We consider two maps associated to this choice of basis.

First, there is an equivariant *augmentation map* $\iota \colon L_m \to L_n$ which attaches a bouquet of $n-m$ circles to the basepoint of each marked graph and marks them with the last $n-m$ basis elements of F_n. This map is simplicial, since a forest collapse has no effect on the bouquet of circles at the basepoint.

Secondly, there is a *restriction map* $\rho \colon K_n \to K_m$ which is easiest to describe using trees. A point in K_n is given by a minimal free simplicial action of F_n on a tree T with no vertices of valence 2. We define $\rho(T)$ to be the minimal invariant subtree for $F_m < F_n$; more explicitly, $\rho(T)$ is the union of the axes in T of all elements of F_m. (Vertices of T that have valence 2 in $\rho(T)$ are no longer considered to be vertices.)

One can also describe ρ in terms of marked graphs. The chosen embedding $F_m < F_n$ corresponds to choosing an m-petal subrose $R_m \subset R_n$. A vertex in K_n is given by a graph Γ marked with a homotopy equivalence $g \colon R_n \to \Gamma$, and the restriction of g to R_m lifts to a homotopy equivalence $\widehat{g} \colon R_m \to \widehat{\Gamma}$, where $\widehat{\Gamma}$ is the covering space corresponding to $g_*(F_m)$. There is a canonical retraction r of $\widehat{\Gamma}$ onto its *compact core*, i.e. the smallest connected subgraph containing all nontrivial embedded loops in Γ. Let $\widehat{\Gamma}_0$ be the graph obtained by erasing all vertices of valence 2 from the compact core and define $\rho(g, \Gamma) = (r \circ \widehat{g}, \widehat{\Gamma}_0)$.

Lemma 2. *For $m < n$, the restriction map $\rho \colon K_n \to K_m$ is simplicial.*

Proof. Any forest collapse in Γ is covered by a forest collapse in $\widehat{\Gamma}$ that preserves the compact core, so ρ preserves adjacency. $\square$

Lemma 3. *For $m < n$, the following diagram of simplicial maps commutes:*

$$\begin{array}{ccc} L_m & \xrightarrow{\iota} & L_n \\ \phi_m \downarrow & & \downarrow \phi_n \\ K_m & \xleftarrow{\rho} & K_n \end{array}$$

Proof. Given a marked graph with basepoint $(g, \Gamma; v) \in L_n$, the marked graph $\iota(g, \Gamma; v)$ is obtained by attaching $n-m$ loops at v labelled by the elements $a_{m+1}, \dots, a_n$ of our fixed basis for F_n. Then $(g_n, \Gamma_n) := \phi_n \circ \iota(g, \Gamma; v)$ is obtained by forgetting the basepoint, and the cover of (g_n, Γ_n) corresponding to $F_m < F_n$ is obtained from a copy of (g, Γ) (with its labels) by attaching $2(n-m)$ trees. (These trees are obtained from the Cayley graph of F_n as follows: one cuts at an edge labelled a_i^{ε}, with $i \in \{m+1, \dots, n\}$ and $\varepsilon = \pm 1$, takes one component of the result, and then attaches the hanging edge to the basepoint v of Γ.) The effect of ρ is to delete these trees. $\square$

4. Proof of the Theorem. In the light of the Corollary and Lemma 3, it suffices to exhibit a sequence of loops ℓ_i in the 1-skeleton of L_3 whose lengths are bounded by a linear

function of i and whose filling area when projected to K_3 grows exponentially as a function of i. Such a sequence of loops is essentially described in [3]. What we actually described there were words in the generators of $Aut(F_3)$ rather than loops in L_3, but standard quasi-isometric arguments show that this is equivalent. More explicitly, the words we considered were $w_i = T^i A T^{-i} B T^i A^{-1} T^{-i} B^{-1}$ where

$$T\colon \begin{cases} a_1 \mapsto a_1^2 a_2 \\ a_2 \mapsto a_1 a_2 \\ a_3 \mapsto a_3 \end{cases} \quad A\colon \begin{cases} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 \\ a_3 \mapsto a_1 a_3 \end{cases} \quad B\colon \begin{cases} a_1 \mapsto a_1 \\ a_2 \mapsto a_2 \\ a_3 \mapsto a_3 a_2 \end{cases}$$

To interpret these as loops in the 1-skeleton of L_3 (and K_3) we note that $A = \lambda_{31}$ and $B = \rho_{32}$ are elementary transvections and T is the composition of two elementary transvections: $T = \lambda_{21} \circ \rho_{12}$. Thus w_i is the product of $8i+4$ elementary transvections. There is a (connected) subcomplex of the 1-skeleton of L_3 spanned by roses (graphs with a single vertex) and Nielsen graphs (which have $(n-2)$ loops at the base vertex and a further trivalent vertex). We say roses are adjacent if they have distance 2 in this graph.

Let $I \in L_3$ be the rose marked by the identity map $R_3 \to R_3$. Each elementary transvection τ moves I to an adjacent rose τI, which is connected to I by a Nielsen graph N_τ. A composition $\tau_1 \dots \tau_k$ of elementary transvections gives a path through adjacent roses $I, \tau_1 I, \tau_1\tau_2 I, \dots, \tau_1\tau_2\dots\tau_k I$; the Nielsen graph connecting σI to $\sigma\tau I$ is σN_τ. Thus the word w_i corresponds to a loop ℓ_i of length $16i+8$ in the 1-skeleton of L_3. Theorem A of [3] provides an exponential lower bound on the filling area of $\phi \circ \ell_i$ in K_3. □

The square of maps in Lemma 3 ought to have many uses beyond the one in this note (cf. [7]). We mention just one, for illustrative purposes. This is a special case of the fact that every infinite cyclic subgroup of $Out(F_n)$ is quasi-isometrically embedded [1].

Proposition. *The cyclic subgroup of $Out(F_n)$ generated by any Nielsen transformation (elementary transvection) is quasi-isometrically embedded.*

Proof. Each Nielsen transformation is in the image of the map $\Phi\colon Aut(F_2) \to Aut(F_n) \to Out(F_n)$ given by the inclusion of a free factor $F_2 < F_n$. Thus it suffices to prove that if a cyclic subgroup $C = \langle c \rangle < Aut(F_2)$ has infinite image in $Out(F_2)$, then $t \mapsto \Phi(c^t)$ is a quasi-geodesic. This is equivalent to the assertion that some (hence any) C-orbit in K_n is quasi-isometrically embedded, where C acts on K_n as $\Phi(C)$ and K_n is given the piecewise Euclidean metric where all edges have length 1.

K_2 is a tree and C acts on K_2 as a hyperbolic isometry, so the C-orbits in K_2 are quasi-isometrically embedded. For each $x \in L_2$, the C-orbit of $\phi_2(x)$ is the image of the quasi-geodesic $t \mapsto c^t.\phi_2(x) = \phi_2(c^t.x)$. We factor ϕ_2 as a composition of C-equivariant simplicial maps $L_2 \xrightarrow{\iota} K_n \xrightarrow{\phi_n} K_2$, as in Lemma 3, to deduce that the C-orbit of $\phi_n\iota(x)$ in K_n is quasi-isometrically embedded. □

A slight variation on the above argument shows that if one lifts a free group of finite index $\Lambda < Out(F_2)$ to $Aut(F_2)$ and then maps it to $Out(F_n)$ by choosing a free factor $F_2 < F_n$, then the inclusion $\Lambda \hookrightarrow Out(F_n)$ will be a quasi-isometric embedding.

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